Lecture 2

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1 Introduction

In differential topology, Morse theory provides techniques to understand the topology of a smooth manifold $\mathcal{M}$ by studying differentiable functions on $\mathcal{M}$. Forman [For98] invented a discrete analogue of Morse theory for cell complexes which we are going to introduce in this lecture. We start by a very simple example from Morse theory to, at least, justify the notations in the discrete version. One may consult the lectures [Mil63] by Milnor, where this example is taken from, for precise definitions and information on Morse theory.

Height Function on Torus. Let $\mathcal{T}$ be a 2-dimensional torus tangent to the xy-plane in point m. Let s_1 and s_2 be the saddle points (critical points of index 1) and M the maximum point (critical point of index 2) of $\mathcal{T}$. See Figure 1. Let $f : \mathcal{T} \to \mathbb{R}$ be the height function, that is to say for $p = (x, y, z) \in \mathcal{T}$ one has $f(p) = z$. For $a \in \mathbb{R}$, let $\mathcal{T}(a)$ be the set of all points p in $\mathcal{T}$ with $f(p) \leq a$.

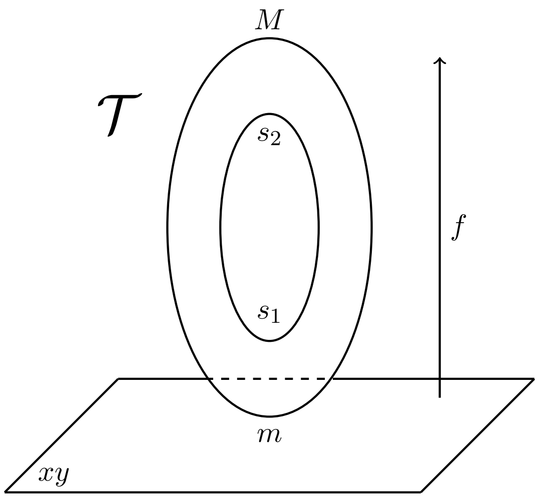

Figure 1: A torus $\mathcal{T}$ tangent to the xy-plane

In particular, $\mathcal{T}(a)$ is empty for $a < 0$ and $\mathcal{T}(0)$ is just one point m. We would like to see how the homotopy type of $\mathcal{T}(a)$ is changing as a increases.

(1) If $a \in (0, f(s_1))$, then $\mathcal{T}(a)$ is a 2-disk and, in particular, homotopy equivalent to a point.

(2) If $a \in (f(s_1), f(s_2))$, then $\mathcal{T}(a)$ is a cylinder and, in particular, homotopy equivalent to a 1-cell attached along its boundary (two points) to a point.

(3) If $a \in (f(s_2), f(M))$, then $\mathcal{T}(a)$ is a torus minus a disk and, in particular, homotopy equivalent to two 1-cells attached along their boundaries to a point.

(4) If $f(M) \leq a$, then $\mathcal{T}(a) = \mathcal{T}$.

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To summarize, we can write:

- If f does not have any critical value in $(a, b]$, then $\mathcal{T}(a) \simeq \mathcal{T}(b)$.
- If f has exactly one critical value in $(a, b]$, then $\mathcal{T}(b)$ is obtained from $\mathcal{T}(a)$ by attaching a cell along its boundary. Moreover, the index of the critical value and the dimension of the new cell are the same.

2 Poset Topology

Abstract Simplicial Complexes. Let V be a finite set. An *abstract simplicial complex* Δ on *vertex set* V is a non-empty collection of subsets of V that is closed under taking subsets. The elements of Δ are called *faces*. The *dimension* of a face is its cardinality minus one and *dimension* of Δ is the maximum dimension of its faces.

Vertex Scheme and Geometric Realization. Let $\mathcal{K}$ be a geometric simplicial complex and $V(\mathcal{K})$ be its vertex set. The *vertex scheme* of $\mathcal{K}$ an abstract simplicial complex Δ together with a bijection $f : V(\Delta) \to V(\mathcal{K})$ in such a way that a subset U of $V(\Delta)$ is in Δ if and only if $\mathrm{conv}(f(U)) \in \mathcal{K}$. If Δ is a vertex scheme of $\mathcal{K}$, we say $\mathcal{K}$ is a geometric realization of Δ.

Lemma 1. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two geometric realization of an abstract simplicial complex Δ, then $\mathcal{K}_1 \cong \mathcal{K}_2$.*

□

Theorem 2. *Every d-dimensional abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$.*

Proof. □

Order Complexes. Let P be a partially ordered set (poset). The *order complex* $\Delta(P)$ is defined to be a simplicial complex whose vertices are elements of P and whose faces are the chains $x_0 < x_1 < \ldots < x_t$ of elements in P.

Face Poset. Let $(\mathcal{X}, \Sigma)$ be a regular cell complex and $\mathcal{K}$ be the set of closed cells (or faces) of $(\mathcal{X}, \Sigma)$. By abuse of language we call $\mathcal{K}$ a regular cell complex. The *face poset* $\mathcal{F}(\mathcal{K})$ is the poset of faces of $\mathcal{K}$ ordered by inclusion. We include the empty set as a face and denote it by $\hat{0}$ in $\mathcal{F}(\mathcal{K})$.

Theorem 3. *A poset P is the face poset of a regular cell complex if and only if $\Delta(\hat{0}, x)$ is homeomorphic to a sphere for all $x \in P$.*

Proof. □

Corollary 4. *Let $\mathcal{K}$ be a regular cell complex. Let σ be a $(d-1)$-face and τ a $(d+1)$-face such that $\sigma < \tau$. Then there are exactly two d-faces λ_1 and λ_2 such that $\sigma < \lambda_i < \tau$.*

3 Discrete Morse Theory

Let $(\mathcal{X}, \Sigma)$ be a regular cell complex and $\mathcal{K}$ be the set of closed cells (or faces) of $(\mathcal{X}, \Sigma)$. Let $f : \mathcal{K} \to \mathbb{R}$ be a real-valued function on faces of $\mathcal{K}$. Let τ be a $(d+1)$-face and σ be a d-face such that $\sigma < \tau$. We say that f has a *descent from σ to τ* if $f(\tau) \leq f(\sigma)$. The set of all descent of f from σ will be denoted by $U_f(\sigma)$ and $L_f(\sigma)$ will denote the set of all descent of f to σ. Clearly, $\sigma \in U_f(\tau)$ if and only if $\tau \in L_f(\sigma)$. We also let $u_f(\sigma)$ (resp. $\ell_f(\sigma)$) denote the cardinality of $U_f(\sigma)$ (resp. $L_f(\sigma)$).

Definition 5 (Discrete Morse Function). A *discrete Morse function* is a 1-1 real-valued function $f : \mathcal{K} \to \mathbb{R}$ such that for all $\sigma \in \mathcal{K}$ one has

$$u_f(\sigma) \leq 1 \quad \text{and} \quad \ell_f(\sigma) \leq 1.$$

A d-face σ is said to be *a d-critical face* (with respect to f) if $u_f(\sigma) = \ell_f(\sigma) = 0$. A *$d$-critical value* of f is the image $f(\sigma)$ of a d-critical face σ. The set of all d-critical faces (w.r.t. f) is denoted by $M_d(f)$ and its cardinality by $m_d(f)$. For $a \in \mathbb{R}$, the *a-level subcomplex* of $\mathcal{K}$ is the set of all faces σ such that there exists $\tau \in \mathcal{K}$ with $\sigma < \tau$ and $f(\tau) \leq a$.

Lemma 6. *If f is a discrete Morse function on $\mathcal{K}$, then $u_f(\sigma) + \ell_f(\sigma) \leq 1$ for all $\sigma \in \mathcal{K}$.*

Proof. Assume not. Then there exist $\tau < \sigma < \lambda$ of consecutive dimensions such that $f(\tau) > f(\sigma) > f(\lambda)$. Now if σ' is the unique other face of $\mathcal{K}$ with $\tau < \sigma' < \lambda$, then $f(\sigma') > f(\tau)$ and $f(\sigma') < f(\lambda)$ (since $u_f(\tau) \leq 1$ and $\ell_f(\lambda) \leq 1$) which is a contradiction. □

Lemma 7. *If f is a discrete Morse function on $\mathcal{K}$, then there exists a critical vertex.*

Proof. $f^{-1}(\min\{f(\sigma) | \sigma \in \mathcal{K}\})$ must be a vertex and is critical. □

Theorem 8. *Let $\mathcal{K}$ be a regular cell complex and f be a discrete Morse function on $\mathcal{K}$.*

(1) *If f does not have any critical value in $(a, b]$, then $\mathcal{K}(a) \nearrow \mathcal{K}(b)$.*

(2) *If f has exactly one d-critical value in $(a, b]$, then $\mathcal{K}(b)$ is obtained from $\mathcal{K}(a)$ by attaching a d-cell along its boundary.*

Proof. □

Corollary 9. *Let $\mathcal{K}$ be a regular cell complex and f be a discrete Morse function on $\mathcal{K}$. Then $\mathcal{K}$ is homotopy equivalent to a CW-complex with exactly $m_d(f)$ d-cells for each d.*

Proposition 10. *$\mathcal{K}$ is collapsible if and only if there exists a discrete Morse function on $\mathcal{K}$ with exactly one critical face.*

References

[For98] Robin Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), no. 1, 90–145.

[Mil63] John Milnor, *Morse theory*, Princeton University Press, Princeton, N.J., 1963.